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Similarity solutions of the two-dimensional unsteady boundary-layer equations

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The method of Lie group transformations is used to derive all group-invariant similarity solutions of the unsteady two-dimensional laminar boundary-layer equations. A new method of nonlinear superposition is then used to generate further similarity solutions from a group-invariant solution. Our results are shown to include all the existing solutions as special cases. A detailed analysis is given to several classes of solutions which are also solutions to the full Navier–Stokes equations and which exhibit flow separation.

1. Introduction

Most existing exact solutions in fluid mechanics are similarity solutions in the sense that the number of independent variables is reduced by one or more. They may be derived by dimensional arguments, by the group-theoretic method, or by the *ad hoc* method of free parameters. Among them the group-theoretic method, which includes the dimensional analysis as a special case, is the most systematic in generating similarity solutions.

For the steady two-dimensional laminar boundary-layer equations, there exist two classes of similarity solutions which can be completely characterized by the external inviscid flow as follows:

Class (*a*) The external inviscid flow $u_e(x) = Ax^n$, where A is a constant. This leads to the Falkner–Skan (1931) solution which includes, as special cases, the famous Blasius (1908) solution when $n = 0$ and the Hiemenz (1911) stagnation-point flow when $n = 1$; the latter is also an exact solution to the full Navier–Stokes equations.

Class (*b*) The external flow $u_e(x) = A\,e^{kx}$. This may also be regarded as the limiting case of Class (*a*) as $n \to \infty$.

For the unsteady two-dimensional laminar boundary-layer equations, only three similarity solutions are known to date. (i) Rayleigh (1911) shear flow: This is generated by starting an infinite flat plate from rest and moving in its own plane with a constant velocity. It is also a solution to the full Navier–Stokes equations. (ii) Glauert (1956) and Rott (1956) flow: The flow results from a transverse oscillation of an infinite flat plate about its own plane normal to a uniform oncoming stream. It, too, is also a solution to the Navier–Stokes equations. (iii) Williams & Johnson (1974) flow: This class of similarity solutions was obtained by using a free-parameter method to reduce the number of independent variables of the governing equations from three to two. The resulting equations were then solved numerically to study the unsteady separation phenomenon of the unsteady linearly retarded flow. It should, however, be pointed out that the boundary-layer equations are not valid when flow separation or reversal occurs and their solution cannot really be used to study the behaviour of flow separation.

In this paper we first use the method of Lie group transformations (see e.g.

Ovsiannikov 1982) to derive all possible group-invariant similarity solutions to the problem of unsteady two-dimensional boundary-layer flow of an incompressible fluid. A new method based on nonlinear superposition is then used to generate further similarity solutions which are not group-invariant. It is shown that our solutions include all the existing solutions as special cases, as well as many new ones. A detailed analysis will be given to those solutions that are also solutions to the full Navier–Stokes equations and that exhibit flow separation.

We note that steady and unsteady separated flows have been a topic of intensive study over the past three decades (see e.g. Hui & Tobak 1989 and the references therein), and that reliable theoretical analysis, numerical computations and proper interpretation of experimental observations all depend crucially on a correct understanding of the behaviour of flow separation. In this regard, exact analytical solutions describing separated flows are especially valuable.

We shall begin by formulating the problem in §2 and then outlining the group-theoretic technique in §3. All the group-invariant solutions to the boundary-layer equations are given and classified in §4. In particular, a class of solutions representing the unsteady separated stagnation-point flow will be studied in detail in §5. Finally, those similarity solutions that are obtained by using the method of nonlinear superposition will be given in §6.

2. Formulation

Consider an unsteady two-dimensional viscous flow of an incompressible fluid over a flat plate with the latter taken as $y = 0$ in the Cartesian (x, y)-coordinates. The corresponding components of velocity are denoted respectively by u and v, the pressure, density and kinematic viscosity of the fluid by p, ρ and ν, and the time by t. The two-dimensional unsteady Navier–Stokes equations are

$$\left.\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0, \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \end{aligned}\right\} \tag{1}$$

For simplicity we set $\rho = 1$ and $\nu = 1$; this amounts to choosing suitable units for length and time. For flow of high Reynolds number, the corresponding unsteady laminar boundary-layer equations are

$$\left.\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0, \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= \frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}+\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u_e}{\partial y} &= 0. \end{aligned}\right\} \tag{2}$$

If the flat plate is stationary the boundary conditions are

$$\left.\begin{aligned} u(x,0,t) &= v(x,0,t) = 0, \\ u(x,y,t) &= u_e(x,t) \quad \text{as} \quad y \to \infty. \end{aligned}\right\} \tag{3}$$

The external inviscid flow $u_e(x,t)$ in (2) is usually obtained from inviscid flow calculations and is related to the pressure p by

$$\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}=-\frac{\partial p}{\partial x}. \tag{4}$$

In addition to the boundary conditions, an initial condition on u, i.e. $u(x,y,0)$, must be prescribed in order to form a well-posed problem. In searching for similarity solutions, the initial condition cannot be prescribed but, instead, is determined (see §3).

3. Application of Lie-group method

3.1. *Group-theoretic method*

The essence of the group-theoretic method (Ovsiannikov 1982) is to find a one-parameter Lie group of transformations which leave the system (2) invariant. This can be achieved by solving the determining equations for the differential operator of the group. Even though the system (2) is nonlinear, the determining equations are always linear and are thus much easier to solve. The group of transformations is then further narrowed down to a sub-group under which the boundary conditions (3) are also invariant. The invariant solutions then satisfy a system of differential equations whose number of independent variables is one less than that of the original system, as ensured by the group theory. By repeating the same procedure, one can further reduce the system of partial differential equations of two independent variables to a system of ordinary differential equations.

In applying the group-theoretic method to system (2), we allow the external flow $u_e(x,t)$ and the initial condition $u(x,y,0)$ to be determined so that similarity solutions exist. It turns out that similarity solutions exist only for certain types of external flow u_e and certain types of initial conditions as seen in §4.

3.2. *General solution to the differential operator*

In order to find all the group-invariant similarity solutions to (2), we first find the most general differential operator $\mathscr{D}$ of the transformation group of the form

$$\mathscr{D}=\mathscr{T}\frac{\partial}{\partial t}+\mathscr{X}\frac{\partial}{\partial x}+\mathscr{Y}\frac{\partial}{\partial y}+\mathscr{U}_e\frac{\partial}{\partial u_e}+\mathscr{U}\frac{\partial}{\partial u}+\mathscr{V}\frac{\partial}{\partial v}, \tag{5}$$

such that it leaves system (2) invariant. The invariance conditions of (2) under (5) yield fifty-nine determining equations for the coefficients $\mathscr{T},\ldots,\mathscr{V}$ as functions of t, x, y, u_e, u and v. The most general solution to these determining equations is found by using a symbolic package written in Maple (Char *et al.* 1988) to be

$$\left.\begin{aligned}\mathscr{T}&=2C_3t+C_1,\\ \mathscr{X}&=[2C_3+C_2]x+g(t),\\ \mathscr{Y}&=C_3y+h(x,t),\\ \mathscr{U}_e&=C_2u_e+g'(t),\\ \mathscr{U}&=C_2u+g'(t),\\ \mathscr{V}&=-C_3v+\frac{Dh}{Dt},\end{aligned}\right\} \tag{6}$$

where $C_i, i = 1,2,3$, are arbitrary constants, $g(t)$ and $h(x,t)$ are arbitrary functions, and D/Dt denotes the material derivative; in particular

$$\frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}.$$

When comparing with the solution of Ovsiannikov (1982), we find that his solution misses the x-dependence in the function h and therefore is not the most general solution to the determining equations. The geometric interpretation of the x-dependence of h will be given in the next subsection.

3.3. *Classification of the symmetry groups*

The Lie algebra of the symmetry group of (2) is spanned by the three vector fields:

$$\left.\begin{aligned} V_1 &= \partial_t, \\ V_2 &= x\partial_x + u_e\partial_{u_e} + u\partial_u, \\ V_3 &= 2t\partial_t + 2x\partial_x + y\partial_y - v\partial_v, \end{aligned}\right\} \tag{7}$$

and the infinite-dimensional subalgebras

$$\left.\begin{aligned} V_g &= g(t)\,\partial_x + g'(t)\partial_{u_e} + g'(t)\,\partial_u, \\ V_h &= h(x,t)\,\partial_y + \frac{Dh}{Dt}\partial_v. \end{aligned}\right\} \tag{8}$$

The corresponding transformations for these vector fields, obtained by exponentiation, are listed below:

$$\left.\begin{array}{ll} V_i & \boldsymbol{x} \to \exp(\epsilon V_i)\,\boldsymbol{x}, \\ V_1 & t \to t+\epsilon, \\ V_2 & (x, u_e, u) \to e^{\epsilon}(x, u_e, u), \\ V_3 & (t,x,y,v) \to (e^{2\epsilon}t, e^{2\epsilon}x, e^{\epsilon}y, e^{-\epsilon}v), \\ V_g & (x,u_e,u) \to (x+\epsilon g(t), u_e+\epsilon g'(t), u+\epsilon g'(t)), \\ V_h & (y,v) \to \left(y+\epsilon h(x,t), v+\epsilon\dfrac{Dh}{Dt}\right). \end{array}\right\} \tag{9}$$

As seen from the vector field V_1 in (9), the system (2) is invariant under translations in time t, hence the reference point in time may be chosen arbitrarily. The vector field V_2 generates a uniform scaling transformation on x, u_e and u, whereas the vector field V_3 generates a non-uniform scaling transformation on t, x, y and v, e.g. contracting v while expanding t, x, and y. The vector field V_g relates two coordinate systems, one of which is moving away from the other horizontally with a velocity $x_0'(t) = \epsilon g'(t)$. Thus invariance under V_g means invariance of solutions of (2) under an arbitrary time-dependent translation motion $x_0(t)$ in the x-direction. Finally, the vector field V_h relates a time-dependent curvilinear coordinate system defined by $(\bar{t},\bar{x},\bar{y}) = (t, x, y+y_0(x,t))$, where $y_0(x,t) = \epsilon h(x,t)$, to the usual orthogonal (t,x,y) coordinate system. The invariance of the solution of (2) under this transformation will be used later to study flow past a deforming solid surface $y_0(x,t)$.

3.4. *Flows related by symmetry groups*

According to (9), if $u_e = f_1(x,t)$, $u = f_2(x,y,t)$ and $v = f_3(x,y,t)$ constitute a solution to (2), then the most general form of the solution that is generated by the transformation group (5) and (6) is

$$\left.\begin{aligned} \bar{u}_e &= \frac{1}{k_2}[f_1(X,T)+x_0'(T)], \\ \bar{u} &= \frac{1}{k_2}[f_2(X,Y,T)+x_0'(T)], \\ \bar{v} &= k_3\left[f_3(X,Y,T)+\left.\frac{Dy_0}{Dt}\right|_{(x=X,\,t=T)}\right], \end{aligned}\right\} \tag{10}$$

where $$T = k_3^2(t-t_0), \quad X = k_2 k_3^2[x-x_0(t)], \quad Y = k_3[y-y_0(x,t)], \tag{11}$$

with k_i being constants and $x_0(t), y_0(x,t)$ being arbitrary functions.

From the above analysis, we arrive at the following conclusions about boundary-layer flows which are related by symmetry groups:

(*a*) *A solution of the boundary-layer equations (2) for a flow past a stationary flat plate can be used to describe the flow past a flat plate moving arbitrarily in its own plane with a suitable change of external flow.* Suppose u_e, u and v constitute a solution to the stationary flat-plate problem, then by specifying $k_2 = k_3 = 1$ and $y_0 = 0$ in (10) the resulting solution $\bar{u}_e$, $\bar{u}$ and $\bar{v}$ describes the flow past a flat plate moving at a velocity $x_0'(t)$. The external flows u_e and $\bar{u}_e$ of the two problems are related by

$$\bar{u}_e(x,t) = u_e(x-x_0(t),t)+x_0'(t). \tag{12}$$

Now, since the motion of the plate can be absorbed into the external flow, we shall use the body-fixed coordinate system in what follows without loss of generality, as it has the advantage over the observer-fixed coordinate system in reducing the number of free functions specifying the problem.

(*b*) *The problem of finding the flow past a deforming solid surface $y = y_0(x,t)$ with a given external flow $u_e(x,t)$ can be solved by considering the boundary-layer flow past a stationary flat plate with the same given external flow.* Since a fluid particle on the surface remains there for all time, we have

$$\frac{D(y-y_0)}{Dt} = 0,$$

or $$v(x,y_0,t) = \frac{Dy_0}{Dt}. \tag{13}$$

If the solid surface is deforming in such a way that its particles have no horizontal motion, then the no-slip boundary condition demands that the fluid particles on the surface also have the same vertical motion and no horizontal motion. Therefore

$$u(x,y_0,t) = 0 \tag{14}$$

in addition to (13). Now if $u_e = f_1(x,t), u = f_2(x,y,t)$ and $v = f_3(x,y,t)$ represent the boundary-layer flow past a stationary flat plate, then by specifying $k_2 = k_3 = 1$ and $x_0 = 0$ in (10), the resulting solution $\bar{u}, \bar{v}$ represents the flow past the deforming solid surface $y_0(x,t)$ with the same external flow $u_e = f_1(x,t)$.

Therefore, without loss of generality we only need to find all the group-invariant similarity solutions of (2) under the condition that the plate is stationary and flat, and this is done in §4.

4. Group-invariant solutions

4.1. *Group invariants and boundary conditions*

With the general solution of the operator of the symmetry group given by (6), all group-invariant solutions of (2) and (3) can be found by solving systems of partial differential equations involving only two independent variables. This is achieved by solving the characteristic equations to find all the invariants of the group, which are then used as new variables.

For our case, the characteristic equations are

$$\frac{\mathrm{d}t}{\mathscr{T}} = \frac{\mathrm{d}x}{\mathscr{X}} = \frac{\mathrm{d}y}{\mathscr{Y}} = \frac{\mathrm{d}u_{\mathrm{e}}}{\mathscr{U}_{\mathrm{e}}} = \frac{\mathrm{d}u}{\mathscr{U}} = \frac{\mathrm{d}v}{\mathscr{V}}, \tag{15}$$

where $\mathscr{T}$, $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{U}_{\mathrm{e}}$, $\mathscr{U}$ and $\mathscr{V}$ are given in (6). For the boundary conditions (3) to be invariant, we must have $h = 0$ and $g =$ constant. Equations (15) are then reduced to

$$\frac{\mathrm{d}t}{2C_3 t + C_1} = \frac{\mathrm{d}x}{(2C_3 + C_2)\,x + g} = \frac{\mathrm{d}y}{C_3 y} = \frac{\mathrm{d}u_{\mathrm{e}}}{C_2 u_{\mathrm{e}}} = \frac{\mathrm{d}u}{C_2 u} = \frac{\mathrm{d}v}{-C_3 v}. \tag{16}$$

To illustrate how to solve the above characteristic equations to find the five invariants, let us assume that $C_3 \neq 0$. With a suitable rescaling and renaming of the free parameters, (16) takes the following form:

$$\frac{\mathrm{d}t}{t - t_0} = \frac{\mathrm{d}x}{(1+q)\,x + g} = 2\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u_{\mathrm{e}}}{q u_{\mathrm{e}}} = \frac{\mathrm{d}u}{q u} = -2\frac{\mathrm{d}v}{v}. \tag{17}$$

Unless otherwise stated, we shall set $t_0 = 0$. The first invariant $X(x,t)$ then corresponds to the integration constant of the following equation:

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{(1+q)\,x + g}.$$

We therefore have

$$X = x t^{-(1+q)} - g \int t^{-(2+q)}\,\mathrm{d}t,$$

or

$$X = \begin{cases} (x - x_0)\, t^{-(1+q)} & \text{when} \quad q \neq -1, \\ x - g \ln t & \text{when} \quad q = -1, \end{cases} \tag{18}$$

where x_0 is a constant which will be set equal to zero, unless otherwise stated.

Similarly, for the second invariant Y we solve

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}t}{2t},$$

to obtain

$$Y = \frac{y}{t^{\frac{1}{2}}}$$

Following the same method as before, we obtain three more invariants

$$\left.\begin{aligned} U_e &= \frac{u_e}{t^q}, \\ U &= \frac{u}{t^q}, \\ V &= t^{\frac{1}{2}}v. \end{aligned}\right\} \tag{19}$$

The new system of partial differential equations then has the following form:

$$\left.\begin{aligned} X &= xt^{-(1+q)}, \quad q \neq -1, \\ Y &= \frac{y}{t^{\frac{1}{2}}}, \\ u_e &= t^q U_e(X), \\ u &= t^q U(X, Y), \\ v &= \frac{V(X, Y)}{t^{\frac{1}{2}}}, \end{aligned}\right\} \tag{20}$$

or

$$\left.\begin{aligned} X &= x - q \ln t, \quad q \text{ an arbitrary constant}, \\ Y &= \frac{y}{t^{\frac{1}{2}}}, \\ u_e &= t^{-1} U_e(X), \\ u &= t^{-1} U(X, Y), \\ v &= \frac{V(X, Y)}{t^{\frac{1}{2}}}. \end{aligned}\right\} \tag{21}$$

The boundary conditions (3) then require

$$\left.\begin{aligned} &U(X, 0) = V(X, 0) = 0, \\ &U(X, Y) = U_e(X) \quad \text{as} \quad Y \to \infty. \end{aligned}\right\} \tag{22}$$

The governing differential equations for U, V and U_e can be easily obtained by substituting (20) and (21) into (2); they are given in (27) and (28).

In this way, all possible forms of the invariants together with the corresponding governing differential equations are classified into six distinct classes, which are given in the next subsection.

4.2. *Classification of group-invariant solutions*

Class I:

$$\left.\begin{aligned} u_e &= U_e(t), \\ u &= U(t, y) + U_e(t), \\ v &= 0, \\ \frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial y^2}, \\ U(t, 0) &= -U_e(t), \\ U(t, y) &= 0 \quad \text{as} \quad y \to \infty. \end{aligned}\right\} \tag{23}$$

Class II:

$$\left.\begin{aligned} u_e &= xU_e(t), \\ u &= xU(t,y), \\ v &= V(t,y), \\ U+\frac{\partial V}{\partial y} &= 0, \\ \frac{\partial U}{\partial t}+V\frac{\partial U}{\partial y}+U^2 &= \frac{\partial U_e}{\partial t}+U_e^2+\frac{\partial^2 U}{\partial y^2}, \\ U(t,0) &= V(t,0) = 0, \\ U(t,y) &= U_e(t) \quad \text{as} \quad y\to\infty. \end{aligned}\right\} \tag{24}$$

Class III:

$$\left.\begin{aligned} X &= x-qt, \\ Y &= y, \\ u_e &= U_e(X), \\ u &= U(X,Y), \\ v &= V(X,Y), \\ \frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} &= 0, \\ (U-q)\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y} &= (U_e-q)\frac{\partial U_e}{\partial X}+\frac{\partial^2 U}{\partial Y^2}, \\ U(X,0) &= V(X,0) = 0, \\ U(X,Y) &= U_e(X) \quad \text{as} \quad Y\to\infty. \end{aligned}\right\} \tag{25}$$

Class IV:

$$\left.\begin{aligned} X &= x\exp(-qt), \\ Y &= y, \\ u_e &= \exp(qt)\,U_e(X), \\ u &= \exp(qt)\,U(X,Y), \\ v &= V(X,Y), \\ \frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} &= 0, \\ (U-qX)\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+qU &= (U_e-qX)\frac{\partial U_e}{\partial X}+qU_e+\frac{\partial^2 U}{\partial Y^2}, \\ U(X,0) &= V(X,0) = 0, \\ U(X,Y) &= U_e(X) \quad \text{as} \quad Y\to\infty. \end{aligned}\right\} \tag{26}$$

Class V:

$$\left.\begin{aligned}
X &= x - q\ln t,\\
Y &= \frac{y}{t^{\frac{1}{2}}},\\
u_{\mathrm{e}} &= \frac{U_{\mathrm{e}}(X)}{t},\\
u &= \frac{U(X,Y)}{t},\\
v &= \frac{V(X,Y)}{t^{\frac{1}{2}}},\\
&\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} = 0,\\
(U-q)\frac{\partial U}{\partial X}+\left(V-\frac{Y}{2}\right)\frac{\partial U}{\partial Y}-U &= (U_{\mathrm{e}}-q)\frac{\partial U_{\mathrm{e}}}{\partial X}-U_{\mathrm{e}}+\frac{\partial^2 U}{\partial Y^2},\\
U(X,0) &= V(X,0) = 0,\\
U(X,Y) &= U_{\mathrm{e}}(X) \quad \text{as} \quad Y\to\infty.
\end{aligned}\right\} \tag{27}$$

Class VI:

$$\left.\begin{aligned}
X &= t^{-1-q}x, \quad q \neq -1,\\
Y &= \frac{y}{t^{\frac{1}{2}}},\\
u_{\mathrm{e}} &= t^{q}U_{\mathrm{e}}(X),\\
u &= t^{q}U(X,Y),\\
v &= \frac{V(X,Y)}{t^{\frac{1}{2}}},\\
&\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} = 0,\\
(U-(1+q)X)\frac{\partial U}{\partial X}+\left(V-\frac{Y}{2}\right)\frac{\partial U}{\partial Y}+qU &= (U_{\mathrm{e}}-(1+q)X)\frac{\partial U_{\mathrm{e}}}{\partial X}+qU_{\mathrm{e}}+\frac{\partial^2 U}{\partial Y^2},\\
U(X,0) &= V(X,0) = 0,\\
U(X,Y) &= U_{\mathrm{e}}(X) \quad \text{as} \quad Y\to\infty.
\end{aligned}\right\} \tag{28}$$

Here q is an arbitrary constant.

From (23)–(28) we observe that

(*a*) Class I is an unsteady shear flow.

(*b*) The solutions of Class I and II also satisfy the full Navier–Stokes equations, as is easily verified by substituting them into (1).

(*c*) The steady boundary-layer equations can be obtained from Class III or Class IV by setting $q = 0$.

(*d*) The solutions of Class V exhibit a logarithmic type behaviour in the similarity variable X, but this type of logarithmic behaviour cannot be found in the corresponding steady flow case. It is interesting to note that such logarithmic behaviour also occurs in the similarity solutions to steady hypersonic flow past slender bodies (Hui 1971) or, equivalently, one-dimensional unsteady flow of a perfect gas.

(*e*) The Class VI solution corresponds to the most general power-law case.

The similarity solution of Williams & Johnson (1974) is for the case where the plate is stationary and the external flow is $u_e(x,t) = u_e(\xi)$, with $\xi = (x+Kt)/(1-Bt)$, K and B being constant. Their solution can be obtained from Class VI by setting $q = 0$, and replacing t with $t-1/B$ and x with $x+K/B$. This amounts to taking $t_0 = 1/B$ in (17) and $x_0 = -K/B$ in (18), and results in

$$X = \frac{x+(K/B)}{t-(1/B)} = -B\xi - K.$$

Although their solution gives rise to flow separation, it does not satisfy the Navier–Stokes equations (1). Therefore, as noted earlier, it is not valid near the point of separation and cannot be used to study flow separation behaviour.

4.3. *Further reduction to ordinary differential equations and exact solutions*

By using group-theoretic method once again, each of the six different systems of partial differential equations of two independent variables given in §4.2 can be further reduced to systems of ordinary differential equations (ODEs) which can then be solved easily. The results of the reduction are listed below:

Class I: The governing equation (23) for this class is the heat equation and the similarity solutions for the heat equation are well known (see e.g. Olver 1986).

Class II: Only in two cases can system (24) be reduced to ODEs. The first case corresponds to $U_e = A > 0$, resulting in

$$\left.\begin{aligned} u_e &= Ax, \\ u &= Axf'(A^{\frac{1}{2}}y), \\ v &= -A^{\frac{1}{2}}f(A^{\frac{1}{2}}y), \end{aligned}\right\} \tag{29}$$

where f satisfies

$$f''' + ff'' + 1 - f'^2 = 0, \quad f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{30}$$

This is the Hiemenz (1911) stagnation-point flow solution. The second case corresponds to $U_e = A/t$, resulting in

$$\left.\begin{aligned} u_e &= \frac{A}{t}x, \\ u &= \frac{A}{t}xf'\left(\frac{y}{t^{\frac{1}{2}}}\right), \\ v &= -\frac{A}{t^{\frac{1}{2}}}f\left(\frac{y}{t^{\frac{1}{2}}}\right), \end{aligned}\right\} \tag{31}$$

where $f(Y)$ satisfies

$$f''' + (Af + \tfrac{1}{2}Y)f'' + (1 - Af')f' + A - 1 = 0, \quad f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{32}$$

This solution will be referred to as the *unsteady separated stagnation-point flow* solution (USSP) and will be studied in detail in the next section.

Class III: In order to obtain non-trivial solutions the constant q must be set to zero, resulting in steady boundary-layer flow. All the existing group-invariant steady boundary-layer flow solutions have been listed in the introduction, and no new group-invariant solutions can be found.

Class IV: The system (26) can be reduced to ODEs in the following two cases only. The first case corresponds to $U_e(X) = AX$, i.e. $u_e = Ax$, and this results in the Hiemenz stagnation-point flow solution. The second case corresponds to $U_e = A$ and yields an exact analytic solution

$$\left.\begin{aligned} u_e &= A\exp(k^4 t), \\ u &= A\exp(k^4 t)\,[1 - \exp(-k^2 y)], \\ v &= 0, \end{aligned}\right\} \tag{33}$$

where k is an arbitrary constant. It is noted that (33) also satisfies the full Navier–Stokes equations (1) with the pressure $p = -k^4 u_e x + p_0$, as can be easily verified.

Class V: For (27) to reduce to ODEs the constant q must be set equal to zero. There are again two distinct cases. The first one corresponds to $U_e = AX$ (or $u_e = Ax/t$) and yields the USSP flow solution (31) and (32). The second case corresponds to $U_e = A$ and yields an exact analytic solution

$$\left.\begin{aligned} u_e &= \frac{A}{t} \\ u &= \frac{AY}{t}\exp(-\tfrac{1}{4}Y^2)\left[C + \int_0^{\frac{Y}{2}} \exp(\xi^2)\,\mathrm{d}\xi\right], \quad Y = \frac{y}{t^{\frac{1}{2}}} \\ v &= 0, \end{aligned}\right\} \tag{34}$$

where C is an arbitrary constant. The quantity u/u_e is plotted in figure 1 for the cases $C = -1, 0$ and 1. It can be shown (Appendix A) that this solution possesses overshooting behaviour for all values of C, i.e. $u > u_e$ for some values of $Y > 0$. Furthermore, reverse flow will also occur when $C < 0$.

We note that even though reverse flow and overshooting behaviour occur, this solution (34) remains valid as it satisfies the Navier–Stokes equations (1) with the pressure $p = (A/t^2)x + p_0$.

Class VI: Again, there exist two cases for which (28) reduces to ODEs. The first case corresponds to $U_e = AX$ (or $u_e = Ax/t$) and yields the USSP flow solution (31) and (32). The second case is a family of unsteady shear flows as given by

$$\left.\begin{aligned} u_e &= Bt^q, \\ u &= Bt^q f\left(\frac{y}{t^{\frac{1}{2}}}\right), \\ v &= 0, \end{aligned}\right\} \tag{35}$$

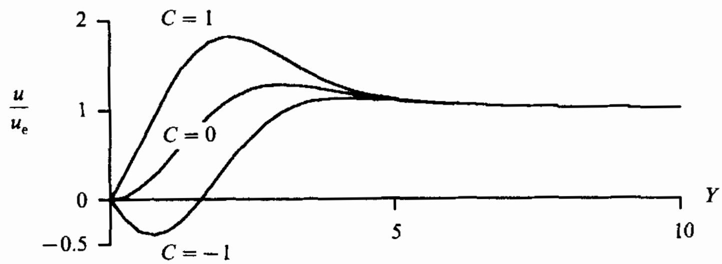

FIGURE 1. u/u_e *vs.* Y.

where $f(Y)$ satisfies

$$f''+\frac{Y}{2}f'-qf+q=0, \quad f(0)=0, \quad f(\infty)=1. \tag{36}$$

When q is a positive integer, (36) has an anlytic solution of the form

$$f(Y)=P(Y)\left[C\int_0^Y \frac{\exp(-\frac{1}{4}\eta^2)}{P^2(\eta)}\,\mathrm{d}\eta-1\right]+1, \tag{37}$$

where

$$\left.\begin{aligned} C&=\left[\int_0^\infty \frac{\exp(-\frac{1}{4}\eta^2)}{P^2(\eta)}\,\mathrm{d}\eta\right]^{-1},\\ P(Y)&=a_0+a_1Y^2+\ldots+a_qY^{2q},\end{aligned}\right\} \tag{38}$$

with

$$\left.\begin{aligned} a_0&=1,\\ a_{i+1}&=\frac{q-i}{2(i+1)(2i+1)}a_i, \quad i=0,\ldots,q-1.\end{aligned}\right\} \tag{39}$$

The convergence of the integrals in (37) and (38) is guaranteed because $a_i>0$ for $i=0,\ldots,q$ as seen from (39). The special case $q=0$ yields the Rayleigh (1911) solution. As a final remark, the solution defined by (35) and (36) also satisfies the Navier–Stokes equations with the pressure

$$p=p_0-qBt^{q-1}x.$$

5. An unsteady separated stagnation-point flow

The following solution, which satisfies both the boundary-layer equations (2) and the Navier–Stokes equations (1), represents an *unsteady separated stagnation point flow* (USSP):

$$\left.\begin{aligned} u_e&=\frac{A}{t}x,\\ u&=\frac{A}{t}xf'\left(\frac{y}{t^{\frac{1}{2}}}\right),\\ v&=-\frac{A}{t^{\frac{1}{2}}}f\left(\frac{y}{t^{\frac{1}{2}}}\right),\\ p&=-\left[\frac{A(A-1)}{2t^2}x^2+\frac{Ayf}{2t^{\frac{3}{2}}}+\frac{A^2f^2}{2t}+\frac{Af'}{t}+p_0\right],\end{aligned}\right\} \tag{40}$$

where $f(Y)$ satisfies the same equation as (32).

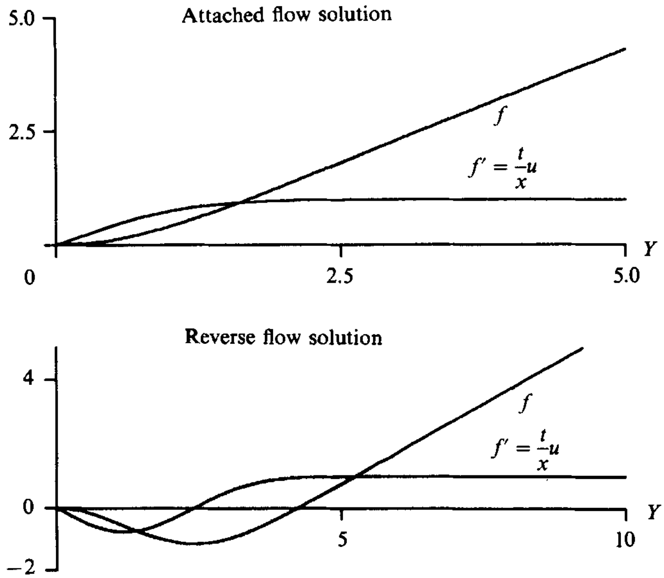

FIGURE 2. Attached- and reverse-flow solutions to (41) for USSP when $A = 1$.

For the special case $A = 1$, these become

$$f''' + (f + \tfrac{1}{2}Y)f'' + (1 - f')f' = 0, \quad f(0) = f'(0) = 0, \quad f'(\infty) = 1, \tag{41}$$

which closely resembles the Hiemenz (1911) steady stagnation-point flow solution (29) and (30). By comparing the USSP flow equation (41) with (30), we find that the unsteady effect arises from the terms $\frac{1}{2}Yf'' + f' - 1$ in (41), as the latter may be written

$$\underbrace{f''' + ff'' + 1 - f'^2}_{\text{Hiemenz}} + \underbrace{\tfrac{1}{2}Yf'' + f' - 1}_{\text{unsteady effect}} = 0.$$

It is interesting to note that in contrast to (30), the solution to (41) is not unique. The results of numerical computations show that two solutions to (41) exist, one representing an attached flow, the other a reverse flow (figure 2).

The stream function Ψ of the flow (40) is given by

$$\Psi = \frac{A}{t^{\frac{1}{2}}} f\left(\frac{y}{t^{\frac{1}{2}}}\right) x.$$

For the reverse-flow case, the streamline patterns are plotted in figure 3 for the case $A = 1$ and for time $t = 1, 2, 3$, where the existence of a saddle point is noted. This clearly represents a separated flow (Hui & Tobak 1989) with $\Psi = 0$ being the separation streamline.

When $A \neq 1$, the situation is more complicated. For all values of A tested, there exist at least two solutions of (32). For some values of A, e.g. $A = 2$, five different solutions of f exist. The corresponding streamlines for the case $A = \frac{1}{2}$ and $A = 2$ are given in figures 4 and 5.

From (40), we deduce that the surface pressure at $y = 0$ reaches an extremum at the origin. It is a minimum if $A(A-1) < 0$ and a maximum if $A(A-1) > 0$, as is easily seen from $\partial p/\partial x$ of (40). The results of numerical experiments on (32) for various values of A suggest that whenever there is flow separation, there is a pressure

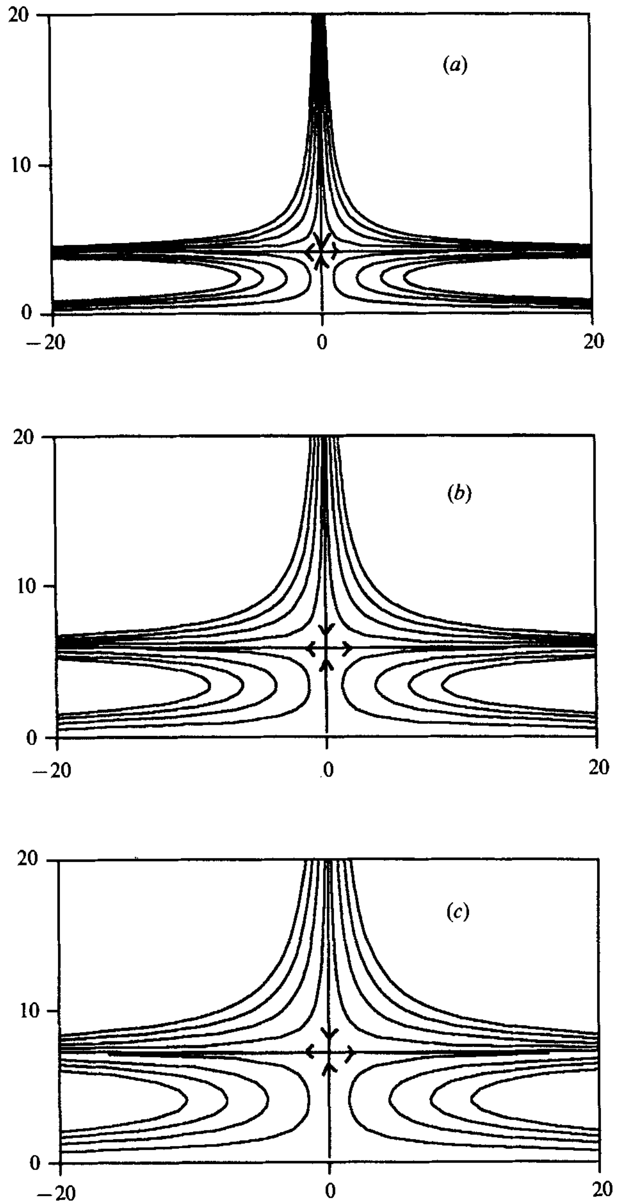

FIGURE 3. Streamline patterns for USSP when $A = 1$. (a) $t = 1$, (b) $t = 2$, (c) $t = 3$.

minimum on the body surface, but the converse is not generally true. This is in agreement with the theoretical prediction of M. Tobak (private communication). On the other hand, flow reattachment will occur when there is a pressure maximum. In this regard, the case $A = 1$ can be thought of as being the borderline case between the solutions that exhibit flow separation (the case where $A < 1$) and those that exhibit only flow reattachment (the case where $A > 1$).

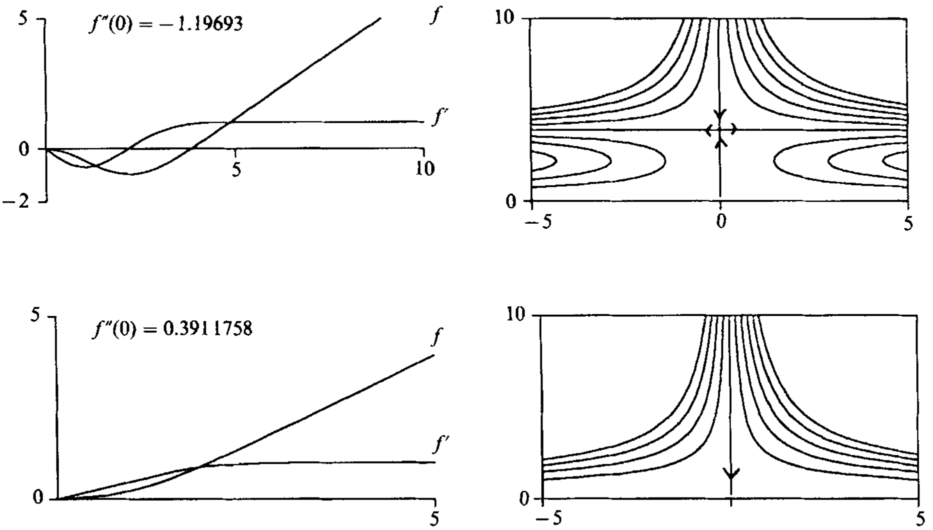

FIGURE 4. The solutions for f when $A = \frac{1}{2}$ together with the corresponding streamlines.

6. Further similarity solutions

In this section, we derive further similarity solutions from some of the basic group-invariant solutions obtained in the last section. We first use the Hiemenz (1911) stagnation-point flow solution as the basic solution to illustrate the method. The method will then be applied to the USSP flow solution to generate further solutions.

6.1. *Generalized Hiemenz stagnation-point flow*

We look for solutions of the boundary-layer equations (2) and boundary conditions (3) of the form

$$\left.\begin{aligned} u_e &= Ax+S(t), \\ u &= Axf'(Y)+\Omega(t,Y), \quad Y = A^{\frac{1}{2}}y, \\ v &= -A^{\frac{1}{2}}f(Y), \end{aligned}\right\} \tag{42}$$

with
$$\Omega(t,0) = 0, \quad \Omega(t,\infty) = S(t), \tag{43}$$

where f is the Hiemenz solution (30). We attempt to determine the functions Ω and S such that (2) reduce to ordinary differential equations.

Let
$$S(t) = \sum_{n=0}^{k} B_n F_n(t), \tag{44}$$

where the B_n are non-zero constants and the F_n are linearly independent functions of time t. Correspondingly, let

$$\Omega(t,Y) = \sum_{n=0}^{k} B_n F_n(t)\, g_n(Y), \tag{45}$$

where the functions g_n are to be determined subject to the conditions that $g_n(0) = 0$ and $g_n(\infty) = 1$, $n = 0,\dots,k$, as required by (43).

Substituting (42)–(45) into (2), we obtain

$$\sum_{n=0}^{k} [B_n F'_n(g_n-1)+AB_n F_n(f'g_n-fg'_n-1-g''_n)] = 0. \tag{46}$$

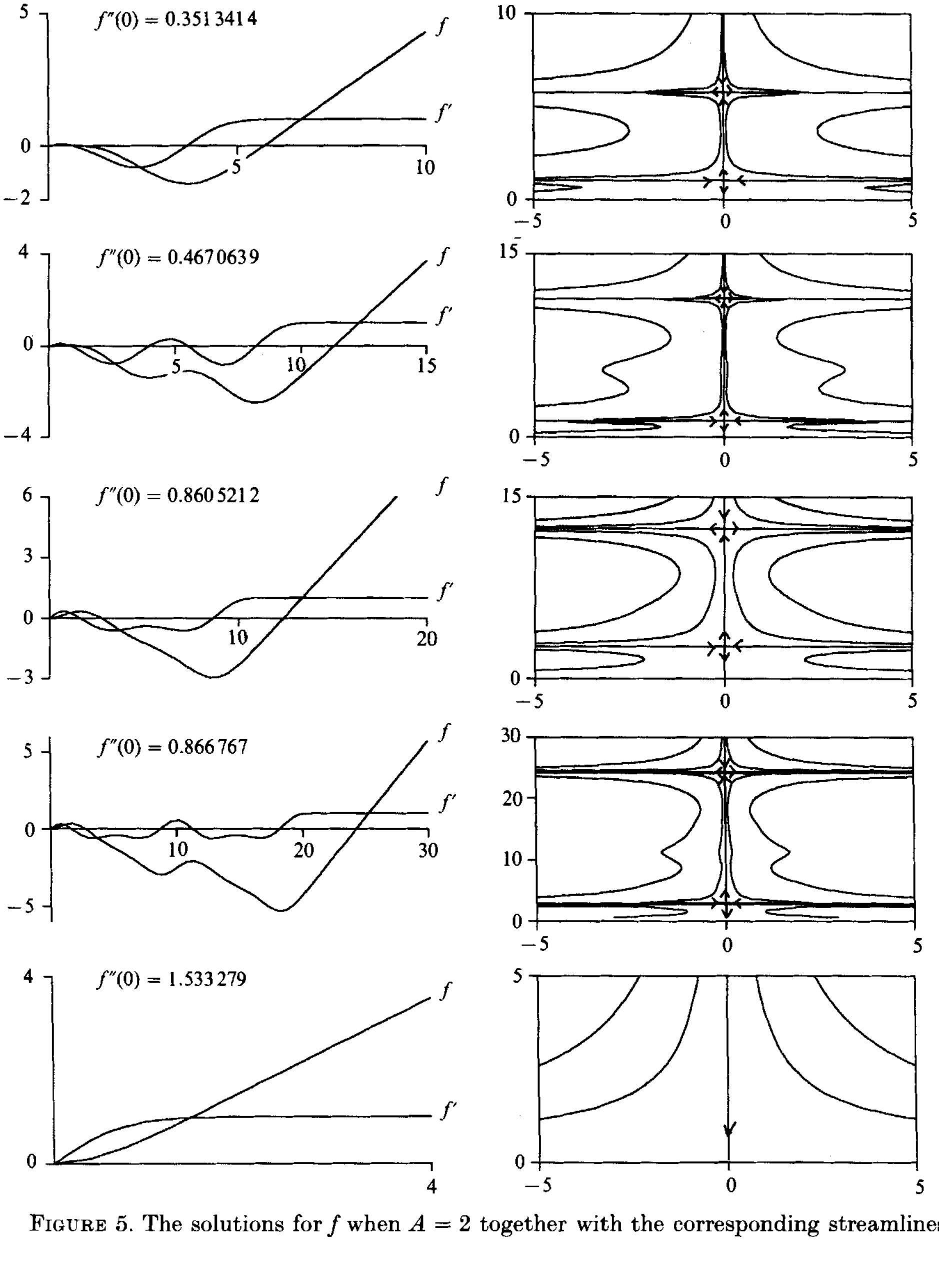

FIGURE 5. The solutions for f when $A = 2$ together with the corresponding streamlines.

Let us assume that for all F_n, $n = 0, \ldots, k$, there exist constants C_{ni}, $i = 0, \ldots, k$, such that

$$F'_n = \sum_{i=0}^{k} C_{ni} F_i. \tag{47}$$

In Appendix B all linearly independent functions F_n satisfying (47) are found to be a suitable linear combination of the following functions

$$y_i(t;\lambda) = \exp(\lambda t) \sum_{j=0}^{i} \frac{t^j}{j!}, \tag{48}$$

where λ is a parameter. As an illustration, the external flow of the Glauert (1956) and Rott (1956) solution, when given in the body-fixed coordinates, is

$$u_e = Ax + B_0 \exp(\mathrm{i}\omega t) \tag{49}$$

It corresponds to a subcase of ours when $F_0 = y_0(t;\mathrm{i}\omega)$, $k = 0$ and $C_{00} = \mathrm{i}\omega$.

Substituting (47) into (46), we obtain

$$\sum_{n=0}^{k}\left[g_n''+fg_n'-f'g_n+1-\sum_{j=0}^{k}\frac{C_{jn}(g_j-1)B_j}{AB_n}\right]AB_nF_n = 0. \tag{50}$$

Since the F_n are linearly independent, all the coefficients of F_n in (50) vanish, and we have

$$g_n''+fg_n'-f'g_n+1 = \sum_{j=0}^{k}\frac{C_{jn}(g_j-1)B_j}{AB_n}, \quad n = 0,\ldots,k, \tag{51}$$

together with the boundary conditions

$$g_n(0) = 0, \quad g_n(\infty) = 1. \tag{52}$$

Equation (51) shows that the g_n are governed by a system of ODEs and that the constants C_{jn} determine the nature of the interactions of the shear flows g_n. Therefore for a set of linearly independent functions F_n that satisfy (47), equations (42)–(45), (51) and (52) constitute a similarity solution to the boundary-layer equations (2) with boundary conditions (3). It can be verified that this solution also satisfies the Navier–Stokes equations (1) with the pressure

$$p = -(\tfrac{1}{2}A^2x^2+SAx+S'x+\tfrac{1}{2}Af^2+Af'+p_0). \tag{53}$$

The stream function Ψ for the flow given by (42) is

$$\Psi = A^{\frac{1}{2}}fx+\frac{\int_0^Y \Omega(t,\eta)\,\mathrm{d}\eta}{A^{\frac{1}{2}}}. \tag{54}$$

For a particular streamline $\Psi = \Psi_0$, we have

$$x = \frac{A^{\frac{1}{2}}\Psi_0-\int_0^Y \Omega(t,\eta)\,\mathrm{d}\eta}{Af}. \tag{55}$$

Since $f''(0)$ is positive for the Hiemenz solution and $v = -A^{\frac{1}{2}}f < 0$, the flow is always reattached and the point of reattachment, where $\partial u/\partial y = 0$, is given by

$$x_a = -\frac{\Omega_Y(t,0)}{Af''(0)}, \tag{56}$$

with $\Psi = 0$ being the reattaching streamline. From (53), we know that at $y = 0$, the pressure reaches a maximum at

$$x_0 = -\frac{SA+S'}{A^2}. \tag{57}$$

The results for this class of flow where the pressure attains a maximum on the body-surface and the flow is always reattached are in agreement with the theoretical predictions of Tobak.

6.2. *Generalized USSP flow solution*

By using a similar method to that shown in §6.1, we derive a generalized USSP flow solution. We look for solutions of the boundary-layer equations (2) and boundary conditions (3) of the form

$$\left.\begin{aligned} u_e &= \frac{Ax}{t}+S(t), \\ u &= \frac{Ax}{t}f'(Y)+\Omega(t,Y), \\ v &= -\frac{A}{t^{\frac{1}{2}}}f(Y), \end{aligned}\right\} \tag{58}$$

where $Y = y/t^{\frac{1}{2}}$, f satisfies (32), and

$$\Omega(t,0) = 0, \quad \Omega(t,\infty) = S(t). \tag{59}$$

We shall determine classes of functions $\Omega(t, Y)$ such that the boundary-layer equations (2) reduce to a system of ordinary differential equations.

Let
$$S(t) = \sum_{n=0}^{k} B_n F_n(t), \tag{60}$$

where the B_n are non-zero constants and the $F_n(t)$ are linearly independent functions of time t. Correspondingly, let

$$\Omega(t,Y) = \sum_{n=0}^{k} B_n F_n(t)\, g_n(Y), \tag{61}$$

where the functions g_n are to be determined subject to the conditions that $g_n(0) = 0$ and $g_n(\infty) = 1$, as required by (59).

Substituting (58), (60) and (61) into (2) and (3), we obtain

$$\Sigma_{n=0}^{k}[B_n F_n[g''_n+(Af+\tfrac{1}{2}Y)g'_n-Af'g_n+A]-B_n(g_n-1)\,tF'_n] = 0. \tag{62}$$

Suppose that for all F_n, there exist constants $C_{ni}, i = 0,\ldots,k$, such that

$$tF'_n = \sum_{i=0}^{k} C_{ni} F_i, \tag{63}$$

then (62) becomes

$$\sum_{n=0}^{k}\left[g''_n+(Af+\tfrac{1}{2}Y)\,g'_n-Af'g_n+A-\sum_{i=0}^{k}\frac{B_i C_{in}(g_i-1)}{B_n}\right]B_n F_n = 0.$$

Since the F_n are linearly independent, we have

$$g''_n+(Af+\tfrac{1}{2}Y)\,g'_n-Af'g_n+A = \sum_{i=0}^{k}\frac{B_i C_{in}(g_i-1)}{B_n}. \tag{64}$$

The boundary conditions for the g_n are

$$g_n(0) = 0, \quad g_n(\infty) = 1. \tag{65}$$

If we let $\tau = \ln t$, then

$$\frac{dF_n}{d\tau} = t\frac{dF_n}{dt} = \sum_{i=0}^{k} C_{ni} F_i.$$

Therefore from the discussion of Appendix B, the existence of the F_i, is guaranteed and the F_i must be in the form

$$F_i = \sum_{j,k} a_{ijk} \exp(\lambda_k \tau)\tau^j = \sum_{j,k} a_{ijk} t^{\lambda_k} \ln^j (t).$$

It is clear from (64) that the constants C_{in} determine how the functions g_i interact with each other. Therefore, for a set of linearly independent functions F_n which satisfy (63), equations (58), (60), (61), (64)–(65) constitute a similarity solution to the boundary-layer equations (2) and (3). It can be easily verified that this solution also satisfies the Navier–Stokes equations (1) with the pressure

$$p = -\left[\frac{A(A-1)}{2t^2}x^2 + S'x + \frac{SA}{t}x + \frac{A}{2t^{\frac{3}{2}}}yf + \frac{A^2f^2}{2t} + \frac{Af'}{t} + p_0\right]. \tag{66}$$

There are several important features of this family of solutions, namely:

(*a*) When $A = 0$, it represents a class of shear flow characterized by the functions g_i.

(*b*) When $B_i = 0$, it reduces to the USSP flow solution.

(*c*) The case $AB_i \neq 0$ is the superposition of the flows in (*a*) and (*b*) in which the shear flows g_i are affected by the USSP flow f via (64), but the shear flows do not affect the USSP flow.

The stream function Ψ for the flow given by (58) is

$$\Psi = \frac{Af}{t^{\frac{1}{2}}}x + t^{\frac{1}{2}}\int_0^Y \Omega(t,\eta)\,d\eta. \tag{67}$$

For a particular streamline $\Psi = \Psi_0$, we have

$$x = \frac{t^{\frac{1}{2}}\Psi_0 - t\displaystyle\int_0^Y \Omega(t,\eta)\,d\eta}{Af(Y)} = \mathscr{S}(\Psi_0, t, Y). \tag{68}$$

Therefore if $\Psi = \Psi_0$ denotes the separation (or reattachment) streamline at time t with x_s (finite) being the point of separation (or reattachment), we must have

$$x_s = \lim_{Y\to 0} \mathscr{S}(\Psi_0, t, Y). \tag{69}$$

Since $f(0) = f'(0) = \Omega(t,0) = 0$, it is clear that $\Psi = 0$ is the only separation (reattachment) streamline with the position of separation (reattachment) given by

$$x_s = -\frac{t\Omega_Y(t,0)}{Af''(0)}. \tag{70}$$

Since $v = -Af/t^{\frac{1}{2}}$ the flow is separated when $f''(0) < 0$ (i.e. $v > 0$ when y is sufficiently small) and reattached when $f''(0) > 0$ (i.e. $v < 0$ when y is sufficiently small). In view of the above analysis, the USSP flow determines the principal behaviour of the flow represented by (58).

From (66), we know that the surface pressure at $y = 0$ reaches an extremum at

$$x_0 = -\frac{t(tS' + AS)}{A(A-1)}, \tag{71}$$

which is dependent on the external flow. It is a minimum when $A(A-1) < 0$ and a

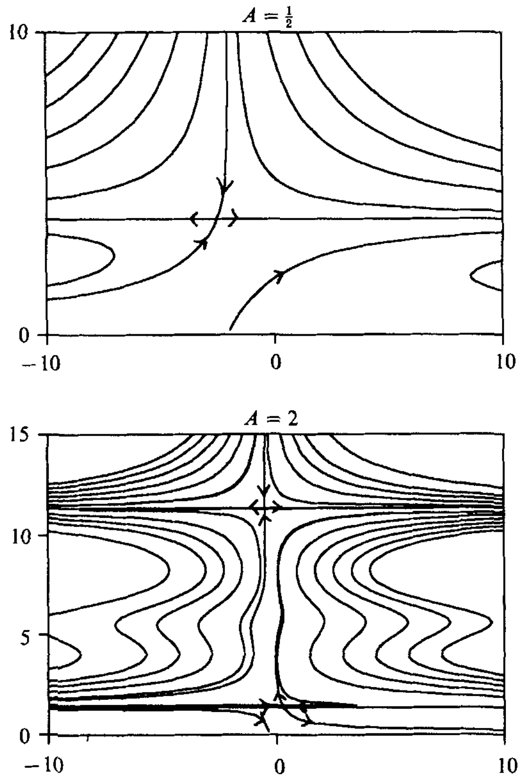

FIGURE 6. Streamline patterns for $S(t) = t^{-2}$ and $A = 0.5, 2$ at $t = 1$.

maximum when $A(A-1) > 0$ and this conclusion holds regardless of the shear flow S. We therefore conclude from the study of the USSP flow that if the flow (58) is separated there must be a pressure minimum on the body surface. On the other hand, flow reattachment will occur if and only if there is a pressure maximum.

Typical streamline patterns showing flow separation ($A = \frac{1}{2}$) and reattachment ($A = 2$) are given in figure 6 in which $S(t) = t^{-2}$ was used.

7. Conclusions

The Lie-group method of symmetry has proved to be a very powerful tool in generating all group-invariant similarity solutions representing two-dimensional unsteady boundary-layer flow over a plate. These solutions are classified into six distinct classes according to their functional forms. More similarity solutions are also generated by using a method of nonlinear superposition of suitably chosen shear flows. We have shown that our solutions include all the previously known solutions as special cases. In addition, many new solutions are found which are also solutions to the full Navier–Stokes equations. Thus these new solutions remain valid even when flow separation or reversal occurs. Many of our new solutions represent separated flows and they have been used in studying the relationship between flow separation (or reattachment) and the pressure extremum on the body surface.

Appendix A

In this Appendix, we show that the flow given by (34) possesses overshooting behaviour for all values of C, i.e. $u > u_e$ for some values of $Y > 0$. We also show that reverse flow will occur when $C < 0$.

To show the overshooting behaviour, let us consider the behaviour of the function

$$F = \frac{u}{u_e} = Y \exp(-\tfrac{1}{4}Y^2)\left[C + \int_0^{\frac{Y}{2}} \exp(\xi^2)\,d\xi\right]$$

near its maximum by examining its derivative

$$\frac{dF}{dY} = (1-\tfrac{1}{2}Y^2)\exp(-\tfrac{1}{4}Y^2)\left[C + \int_0^{\frac{Y}{2}} \exp(\xi^2)\,d\xi\right] + \tfrac{1}{2}Y.$$

At a stationary point $dF/dY = 0$, hence

$$C = \frac{Y}{Y^2-2}\exp(\tfrac{1}{4}Y^2) - \int_0^{\frac{Y}{2}} \exp(\xi^2)\,d\xi \tag{A 1}$$

and the corresponding value of F is

$$F = \frac{Y^2}{Y^2-2}. \tag{A 2}$$

We now show that for any given constant C, there exists a $Y > 0$ such that (A 1) is satisfied. To see this, we note that, from (A 1),

$$\frac{\partial C}{\partial Y} = -\frac{4\exp(\tfrac{1}{4}Y^2)}{(Y^2-2)^2} < 0.$$

Accordingly $\partial C/\partial Y \to -\infty$ as $Y \to +\infty$, so we have $C \to -\infty$ as $Y \to +\infty$. Moreover, $C \to +\infty$ as $Y \downarrow \sqrt{2}$ as seen from (A 1). Therefore (A 1) defines a map C which maps the interval $(\sqrt{2}, \infty)$ to the entire real line. So for any given C, there exists at least one $Y > \sqrt{2}$ satisfying (A 1). We therefore conclude from (A 2) that for any given value of C, there exists a Y such that $F = u/u_e > 1$, i.e. the solution (34) exhibits an overshooting behaviour for any value of C.

Furthermore, if $C < 0$, there exists a $Y > 0$ such that

$$C + \int_0^{\frac{Y}{2}} \exp(\xi^2)\,d\xi < 0,$$

and so $u < 0$. This is to say that reverse flow will always occur whenever $C < 0$.

Appendix B

In this Appendix, we classify all the linearly independent functions $F_n(t), n = 0, \ldots, k$, that satisfy

$$F'_n = \sum_{i=0}^{k} C_{ni} F_i. \tag{B 1}$$

When written in matrix notation, (B 1) becomes

$$\boldsymbol{F}' = \boldsymbol{CF} \tag{B 2}$$

where
$$\boldsymbol{F} = \begin{bmatrix} F_0 \\ \vdots \\ F_n \end{bmatrix}$$

and $\boldsymbol{C} = [C_{ij}]$. Suppose $\boldsymbol{J}$ is the Jordan matrix of $\boldsymbol{C}$, then there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{C} = \boldsymbol{PJP}^{-1}$. If $\boldsymbol{Y}$ satisfies $\boldsymbol{Y}' = \boldsymbol{JY}$, then it can be verified that $\boldsymbol{F} = \boldsymbol{PY}$ is a solution to (B 2). Since $\boldsymbol{P}$ is invertible, the F_i are linearly independent if and only if the Y_i are linearly independent. Let $\boldsymbol{J}$ be

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_0 & & \\ & \ddots & \\ & & \boldsymbol{J}_s \end{bmatrix},$$

where
$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & \lambda_i \end{bmatrix},$$

and let $\boldsymbol{Y}_{\lambda_i}$ be a solution to $\boldsymbol{Y}' = \boldsymbol{J}_i \boldsymbol{Y}$, then

$$\boldsymbol{Y}_\lambda = \begin{bmatrix} \boldsymbol{Y}_{\lambda_0} \\ \vdots \\ \boldsymbol{Y}_{\lambda_s} \end{bmatrix}$$

is a solution to $\boldsymbol{Y}' = \boldsymbol{JY}$. Therefore, without loss of generality, we need only consider the case where

$$\boldsymbol{J}_\lambda = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & \lambda \end{bmatrix}.$$

It can then be shown that

$$y_i(t;\lambda) = \exp(\lambda t) \sum_{j=0}^{i} \frac{t^j}{j!} \tag{B 3}$$

is a solution to $\boldsymbol{Y}' = \boldsymbol{J}_\lambda \boldsymbol{Y}$ and that the y_i are linearly independent. We therefore conclude that

$$\boldsymbol{F} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{Y}_{\lambda_0} \\ \vdots \\ \boldsymbol{Y}_{\lambda_s} \end{bmatrix}, \tag{B 4}$$

where $\boldsymbol{P}$ is an invertible matrix and $\boldsymbol{Y}_\lambda = [y_j(t;\lambda)]$, is the most general form of linearly independent functions that satisfy $F'_i = \Sigma C_{ij} F_j$ for some constants C_{ij}.

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